Asymptotic freedom, non-Gaussian perturbation theory, and the application of renormalization group theory to isotropic turbulence

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A model equation with an extended range of asymptotic freedom is used as the basis of a non-Gaussian perturbation expansion in powers of the control parameter of the conditional average. Re-expansion in the local Reynolds number allows the systematic rederivation of an earlier, heuristic theory [W. D. McComb and A. G. Watt, Phys. Rev. Lett. **65**, 3281 (1990)].

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Real fluid turbulence satisfies the Navier-Stokes equations (NSEs) and has a well established phenomenology, which is partially summarized in the form of the energy spectrum shown in Fig. 1. In the context of renormalization group (RG), the local (in wave number) Reynolds number $R_0(k) \sim [E(k)]^{1/2}/\nu_0 k^{1/2}$ (see Ref. [1]), where ν_0 is the fluid kinematic viscosity, is the coupling constant. From Fig. 1, we see that the local coupling vanishes at $k=0$ and as $k \to \infty$. In field theory this is described as asymptotic freedom [2]. In turbulence it is present in both the "ultraviolet" and the "infrared."

Before putting forward our proposal for a theoretical model with associated perturbation expansion, we will first expand on these preliminary remarks, introducing both the turbulence problem and its relationship to statistical field theory. To state the problem, we consider a random velocity field $u_\alpha(\mathbf{k},t)$, with zero mean, on the wave number range $0 \leqslant k \leqslant k_{\max}$; and with covariance given by

$$\langle u_\alpha(\mathbf{k},t) u_\beta(\mathbf{k}',t') \rangle = P_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k}+\mathbf{k}') Q(k,t-t'), \qquad (1)$$

where $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |k|^{-2}$. As is well known, this form is appropriate to turbulence which is homogeneous, isotropic, and stationary in time.

In order to pose this problem, we must provide some sort of input in order to sustain the turbulence against the effects of viscous dissipation. This can be done in various ways, but the most usual is to introduce a stirring force $f_\alpha(\mathbf{k},t)$, with covariance

$$\langle f_\alpha(\mathbf{k},t) f_\beta(\mathbf{k}',t') \rangle = P_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k}+\mathbf{k}') D(k), \qquad (2)$$

where the force spectral density $D(k)$ is arbitrarily chosen and determines the rate ε_W at which the forces do work on the fluid. This is readily shown to be

$$\varepsilon_W = \int_0^\kappa 4\pi k^2 D(k) dk \equiv \int_0^\kappa W(k) dk, \qquad (3)$$

where W is the force spectrum and κ is chosen to be small compared to the wave numbers characterized by inertial (i.e., nonlinear) transfer, which in turn are small compared to those wave numbers where the viscous dissipation occurs. In other words, the effects of the force spectrum should be confined to small wave numbers so that the energy spectrum at large wave numbers is determined solely by the equation of motion. Also note that for stationary turbulence we must have $\varepsilon_W = \varepsilon$, where ε is the viscous dissipation.

It is not possible to give a precise value for $k_{\max}$, so we adopt the expedient of choosing it large enough to capture the rate of dissipation ε to some specified accuracy. If the Reynolds number is sufficiently large, then we expect to observe a $k^{-5/3}$ spectrum over a subrange of wave numbers $k_b \leqslant k \leqslant k_t$. The lower limit k_b is set by the autocorrelation of the stirring forces while the upper limit k_t is related to the Kolmogorov dissipation wave number $k_d^{(0)} = (\varepsilon/\nu_0^3)^{1/4}$, although $k_d^{(0)}$ overestimates k_t by nearly an order of magnitude and underestimates $k_{\max}$ by a similar amount.

Referring now to Fig. 1, we reflect conventional wisdom in dividing the energy spectrum into a low-wave-number region, where the shape of the spectrum depends on the way in which it was produced (the "production range") and the higher wave numbers where we may expect universal behavior with the spectrum taking a form $g(k/k_d^{(0)})$, where g is some well-behaved function which remains to be determined. The universal range can be further divided into the inertial range, where $E(k) = \alpha \varepsilon^{2/3} k^{-5/3}$, and the viscous range, where the spectrum falls off faster than a power. The prefactor α is often referred to as the Kolmogorov constant and is

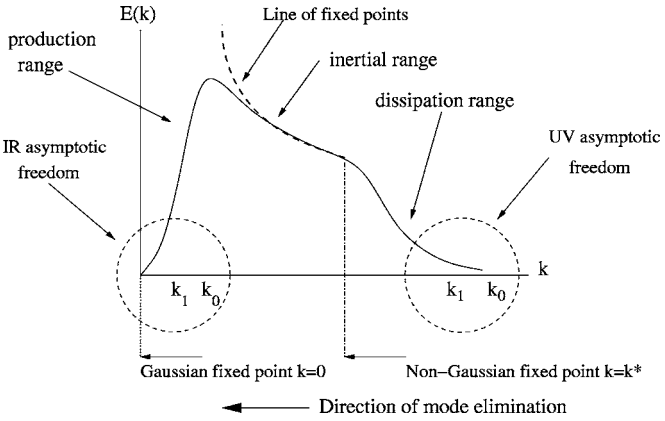

FIG. 1. Sketch of the turbulence energy spectrum $E(k)$.

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currently believed to take a value of about $\alpha=1.62$.

The development of modern turbulence theory was based on an analogy between the equations for macroscopic fluid turbulence and those of quantum field theory [3–6]. In particular, it was pointed out that turbulence provided an example of a quantum field theory in which the coupling constant (i.e., the Reynolds number) could be varied in the laboratory from zero to infinity [7]. Later, it was argued that an analogy existed between turbulence and critical phenomena [8] and this led to the application of RG to stirred hydrodynamics.

However, it must be borne in mind that there are significant differences between the two subjects. In turbulence the relevant power law (Kolmogorov) is completely determined by energy conservation and scale invariance, whereas in critical phenomena there is no analog of the flux of energy though the modes and the power laws must be obtained using RG. In contrast, RG can only verify the power law in turbulence but can predict the prefactor. Similarly, asymptotic freedom in critical phenomena has to be discovered by the vanishing of the coupling constant under RG transformation. Evidently, in turbulence this property emerges trivially by inspection of the spectrum. Nevertheless, we would emphasize the remarkable resemblence between the phenomenology of the turbulence spectrum and a symmetry-breaking phase transition such as paramagnetism $\rightleftharpoons$ ferromagnetism. At $k=0$ and $k\to\infty$ we have trivial fixed points corresponding to, respectively, the high-temperature (disordered) and low-temperature (ordered) fixed points. At $k=k^*=k_t$ we have a nontrivial fixed point marking the onset of scaling behavior, analogous (perhaps) to spontaneous magnetization. Indeed, every wave number in the inertial range corresponds to a fixed point and it is usual to refer to this as "a line of fixed points."

In order to introduce the RG transformation, we filter the velocity field (using a Heaviside step function) at some $k_1<k_0$, and attempt to average out the effect of the high-k modes in the shell $k_1<k<k_0$ (the u^+) while holding the low-k modes (the u^-) constant. There are two difficulties here: first we need a conditional average; second this procedure will generate higher-order nonlinearities, starting with the triplet $u^-u^-u^-$.

We may now state the RG algorithm as applied to the NSE, as follows.

Filter the velocity field $\mathbf{u}(\mathbf{k},t)$ into $\mathbf{u}^-(\mathbf{k},t)$ on $0\leqslant k\leqslant k_1$ and $\mathbf{u}^+(\mathbf{k},t)$ on $k_1\leqslant k\leqslant k_0$. The RG algorithm consists of two steps.

(1) Solve the NSE on $k_1\leqslant k\leqslant k_0$. Substitute that solution for the mean effect of the high-k modes into the NSE on $0\leqslant k\leqslant k_1$. This results in an increment to the viscosity $\nu_0\to\nu_1=\nu_0+\delta\nu_0$.

(2) Rescale the basic variables so that the NSE on $0\leqslant k\leqslant k_1$ looks similar to the original NSE on $0\leqslant k\leqslant k_0$.

These steps are repeated for $k_2<k_1$, $k_3<k_2$, and so on; until a fixed point is reached and this defines the renormalized viscosity. The general idea is illustrated schematically in Fig. 2.

As may be seen from Fig. 1, there are two possible approaches to applying RG to the Navier-Stokes equation. The first of these is to choose the cutoff k_0 at low wave numbers and look for a fixed point as $k\to 0$. In the first version of this approach [9], the initial wave number was taken to be $k_0=\Lambda$, where Λ is an artificial "ultraviolet" cutoff, chosen to be low enough to exclude the effects of the nonlinear cascade. The lower wave number which defines the band of wave numbers to be eliminated is then set as $k_1=\Lambda\exp(-b)$. The situation is illustrated in Fig. 2 and this choice hints at the fact that this approach leads to differential equations rather than finite recursion relations. Next, the u^+ are expanded about a zero order solution obtained from the (bare) viscous response and the stirring forces. The result is a perturbation series, very similar to those in statistical field theory, with coefficients evaluated as integrals against the Gaussian distribution. Accordingly the need for a conditional average is satisfied trivially at each order in perturbation theory and it turns out that the higher-order nonlinearities vanish as $k\to 0$. Corrections to the bare viscosity (and other bare quantities) are calculated in terms of the arbitrarily chosen stirring forces. For example, the viscosity increment $\delta\nu_0$ is given by its value at $k=0$, thus

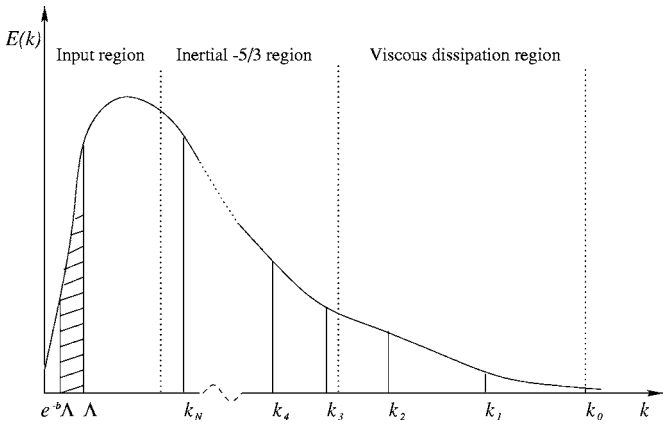

FIG. 2. Sketch illustrating the choice of wave number bands for Gaussian perturbation theory at small wave numbers and the choice of wave number bands for recursive RG at large wave numbers.

$$\delta\nu_0=\frac{\lambda^2 D_0 A(d) S_d[\exp(\epsilon l)-1]}{\nu_0^2\Lambda^\epsilon(2\pi)^d\epsilon}, \tag{4}$$

with $\epsilon=4+y-d$ and

$$A(d)=\frac{d^2-d-\epsilon}{2d(d+2)}, \quad S_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{5}$$

where D_0 and y together represent the effect of the force spectrum, which is arbitrarily chosen to have the form $D(k)=D_0k^{-y}$ and, as we have seen, b is used in defining the cutoff between the u^+ and u^- modes, $\Lambda\exp(-b)$, where Λ is the upper cutoff wave number. Due to the requirement $k\to 0$, this theory does not describe actual turbulence, but is instead a theory of the long-wavelength properties of stirred hydrodynamics. For the pioneering papers, see Refs. [9,10], and for an overview of the many other versions of this approach see Ref. [11].

The second approach is to choose k_0 as the maximum wave number $k_{\max}$, and the lower limit of the band to be

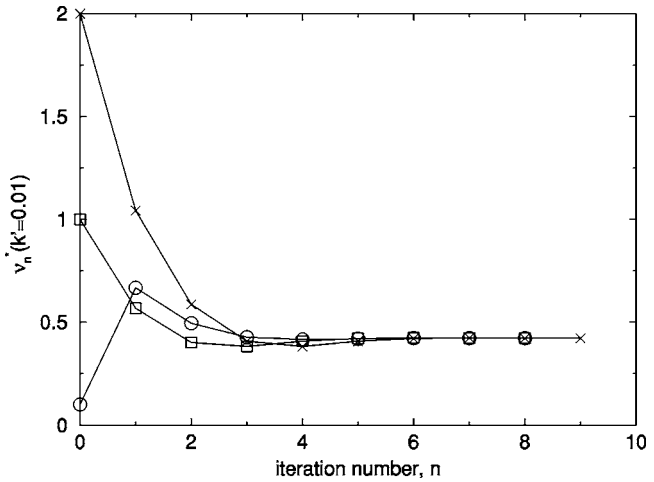

FIG. 3. The renormalized viscosity reaching a fixed point in recursive RG for three different choices of initial (bare) kinematic viscosity.

eliminated is then chosen to be $k_1=hk_0$, where h is the spatial rescaling factor, such that $0\leqslant h\leqslant 1$ and is related to the bandwidth η of the shell to be eliminated by $h=1-\eta$. Further bands can be generated by taking $k_n=h^n k_0$ and the general situation is again illustrated in Fig. 2. This approach rules out Gaussian perturbation theory, as we are now operating in the non-Gaussian, turbulent range of wave numbers. Accordingly one is faced with the need for a nontrivial conditional average. While some early attempts were made to approximate this by a filtered ensemble average [12,13], a more successful approach was to formulate a conditional average which could be evaluated approximately [14–16]. In Fig. 3 we illustrate the behavior of a finite recursion relation for the viscosity, showing that a fixed point is reached after the elimination of four or five shells of modes, independently of a wide variation in choices of initial viscosity. Figure 3 also shows that once a fixed point is reached, the iteration stays there (naturally) even though we are moving to progressively lower wave numbers. As shown in Fig. 1, these fixed points lie on a line following $k^{-5/3}$. This also illustrates the asymptotic nature of our theory, which cannot tell us anything about the production range of wave numbers unless we take a realistic forcing term into account in the equation of motion.

In this work we find that the renormalized viscosity for the Nth cycle (at the fixed point $n=N$), has the similarity solution

$$\nu_N(k) = \alpha^{1/2}\varepsilon^{1/3}k_N^{-4/3}\tilde{\nu}_N(k/k_N), \tag{6}$$

where the function $\tilde{\nu}_N$ is determined from the numerical calculation and is essentially constant as $k\to 0$, with a slight rolloff as k tends to the current cutoff value k_N [see Eq. (82) in Ref. [16]]. This result corresponds to the energy spectrum taking the Kolmogorov form. When this form is substituted for the missing "implicit" stresses in the equation for the explicit modes [i.e., Eq. (10) in this paper], and the resulting equation integrated with respect to wave

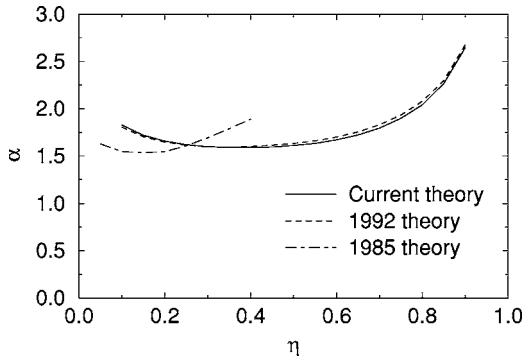

FIG. 4. Dependence of the predicted Kolmogorov constant α on the choice of shell thickness (or bandwidth) parameter η.

number on $0\leqslant k\leqslant k_N$, we obtain the renormalized equation for the dissipation rate as

$$\varepsilon = \int_0^{k_N} 2\nu_N(k)k^2E(k)dk. \tag{7}$$

Then, substitution of the above form for $\nu_N(k)$ and the Kolmogorov form for $E(k)$ yields an expression for the Kolmogorov prefactor

$$\alpha = \left\{2\int_0^1 \tilde{\nu}_N(k')k'^{1/3}dk'\right\}^{-2/3}. \tag{8}$$

These results appear as Eqs. (91) and (92), respectively, in Ref. [16].

Taking a prediction for the Kolmogorov prefactor as a test of the method, we can see from Fig. 4, that our 1992 theory represented quite an improvement on our earlier approach. But this work still relied on "plausible" approximations that higher moments of the high-k modes could be neglected and that their time evolution could be treated by a Markovian approximation. Later we showed that both these approximations could be subsumed into a consistent perturbation expansion in powers of the local Reynolds number; and criticisms of our Taylor series expansion of the chaotic k-space velocity field were answered by a modified procedure which led to a similar expansion of the covariance instead [17]. More recently a minor error in that work was corrected and Fig. 4 shows the current theoretical prediction for α [18]. (Although the vast bulk of work in this field consists of variations on Gaussian perturbation theory [11], it is perhaps worth mentioning that there has been growing recognition of the need to consider a nontrivial average in order to treat real fluid turbulence [19–24].)

However, a price paid for these improvements was that the vanishing of the $2u^-u^+$ term in Eq. (10) under conditional average seemed anomalous when one considered that this term had the same value of local Reynolds number (i.e., coupling parameter) as the u^+u^+ term which made the sole contribution to the renormalized viscosity. In this paper we introduce a non-Gaussian perturbation expansion in the control parameter of the conditional average and then re-expand in the local Reynolds number, hence recovering our existing results in a less heuristic way. The method has its roots in a

previous attempt to produce a more systematic derivation [25] and also in the generalized Reynolds averaging used in the earliest version of this theory [12].

The NSE may be written as

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)u_\alpha(\mathbf{k},t)=M_{\alpha\beta\gamma}(\mathbf{k})\int d^3j u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t), \tag{9}$$

where $M_{\alpha\beta\gamma}(\mathbf{k})=(2i)^{-1}[k_\beta P_{\alpha\gamma}(\mathbf{k})+k_\gamma P_{\alpha\beta}(\mathbf{k})]$. It should be emphasized that we are now operating at wave numbers greater than κ and accordingly the turbulence is independent of the details of the stirring forces and merely depends on the rate at which they do work on the fluid in order to have a stationary state. In order to underline the fact that our perturbation theory, unlike that of Forster *et al.* [10], does not rely on the stirring forces as such, we simply omit them from the equation of motion. Nevertheless, it should be borne in mind that we still require, in principle at least, some input term, be it stirring forces, negative damping or direct injection of energy or momentum.

Let us decompose the NSE into low and high wave number forms. It will also be convenient to introduce an integrating factor and integrate over time, thus

$$u_\alpha^-(\mathbf{k},t)=\lambda\int_{-\infty}^{t}dse^{-\nu_0k^2(t-s)}M_{\alpha\beta\gamma}^-(\mathbf{k})\int d^3j[u_\beta^-(\mathbf{j},s)u_\gamma^-(\mathbf{k}-\mathbf{j},s)+2u_\beta^-(\mathbf{j},s)u_\gamma^+(\mathbf{k}-\mathbf{j},s)+u_\beta^+(\mathbf{j},s)u_\gamma^+(\mathbf{k}-\mathbf{j},s)] \tag{10}$$

and

$$u_\alpha^+(\mathbf{k},t)=\lambda\int_{-\infty}^{t}dse^{-\nu_0k^2(t-s)}M_{\alpha\beta\gamma}^+(\mathbf{k})\int d^3j[u_\beta^-(\mathbf{j},s)u_\gamma^-(\mathbf{k}-\mathbf{j},s)+2u_\beta^-(\mathbf{j},s)u_\gamma^+(\mathbf{k}-\mathbf{j},s)+u_\beta^+(\mathbf{j},s)u_\gamma^+(\mathbf{k}-\mathbf{j},s)], \tag{11}$$

where λ is a book-keeping parameter which is set equal to unity at the end of the calculation. Note that although λ is not renormalized, the local Reynolds number which is the true coupling constant contains the viscosity and is automatically renormalized with it.

A detailed formulation of the conditional average for chaotic systems has been given elsewhere [15]. Here it is sufficient to summarize it as follows. Consider a set of realizations $\{u_\alpha(\mathbf{k},t)\}$, with their low-$k$ parts clustering around one particular member of the set $v_\alpha^-(\mathbf{k},t)$, such that

$$u_\alpha^-(\mathbf{k},t)=v_\alpha^-(\mathbf{k},t)+\phi_\alpha^-(\mathbf{k},t), \tag{12}$$

where ϕ^- is the control parameter for the conditional average and is chosen such that

$$\langle\phi^-\rangle_c=0 \quad \text{and} \quad \langle u^-\rangle_c=v^-. \tag{13}$$

In principle, the bounds on ϕ^- can be determined from a predictability study of Eq. (11), but clearly the more unpredictable turbulence is, the smaller is ϕ^-; and, of course, where the conditional average is over modes with asymptotic freedom, $\phi^-=0$. In this work we shall assume that ϕ^- is small in regions of interest, such that we can treat its square as being of the second order of small quantities and neglect it. For example,

$$\langle u_k^-u_j^-\rangle_c=v_k^-v_j^-+O(\langle\phi_k^-\phi_j^-\rangle_c), \tag{14}$$

as ϕ^- and v^- are not correlated.

For perturbation theory we require a zero-order field, so let us now introduce a function $\tilde{v}_\alpha^+(\mathbf{k},t)$, which is the solution of the model equation

$$\tilde{v}_\alpha^+(\mathbf{k},t)=\lambda\int_{-\infty}^{t}dse^{-\nu_0k^2(t-s)}M_{\alpha\beta\gamma}^+(\mathbf{k})\int d^3j[v_\beta^-(\mathbf{j},s)v_\gamma^-(\mathbf{k}-\mathbf{j},s)+\tilde{v}_\beta^+(\mathbf{j},s)\tilde{v}_\gamma^+(\mathbf{k}-\mathbf{j},s)]. \tag{15}$$

We should emphasize that this equation does not affect the v^- or v^+ fields in any way. We also note that Eq. (15) is not intended to represent any real system, any more than the Gaussian model in the theory of critical phenomena, which possesses neither a low-temperature fixed point nor a phase transition in under four dimensions. Comparing this model equation with Eq. (11), we note the absence of the direct coupling (or "cross") term of the form $2u^-u^+$, means that direct coupling between the low-k and high-k modes is limited to the band of wave numbers $k_1\leqslant k\leqslant 2k_1$. In our present type of calculation the cutoff can be chosen as low as $k_1=0.1k_0$, thus holding out the possibility of an extended range of asymptotic freedom over as much as $0.2k_0\leqslant k\leqslant k_0$.

In the interests of conciseness, we shall give most of our exposition in terms of a contracted notation in which wave number, tensor index, and time are all combined into a single subscript and (for example) the NSE, as given by Eq. (9), becomes

$$L_{0k}u_k=M_ku_ju_{k-j} \quad \text{or} \quad u_k=G_{0k}M_ku_ju_{k-j}, \tag{16}$$

where $G_{0k}=L_{0k}^{-1}$. Now introduce a perturbation series for the u^+ field

$$u_k^+=u_k^{+(0)}+\lambda u_k^{+(1)}+\lambda^2u_k^{+(2)}+\cdots, \tag{17}$$

where λ is the book-keeping parameter. In order to obtain the coefficients, first subtract Eq. (15) from Eq. (11) for u^+ to obtain

$$u_k^+=\tilde{v}_k^++\lambda G_{0k}M_k^+\times\{u_j^-u_{k-j}^-+2u_j^-u_{k-j}^++u_j^+u_{k-j}^+-v_j^-v_{k-j}^--\tilde{v}_j^+\tilde{v}_{k-j}^+\}; \tag{18}$$

then, substituting Eq. (17) into both sides of Eq. (18) yields

$$u_k^{+(0)}+\lambda u_k^{+(1)}+\lambda^2u_k^{+(2)}+\cdots=\tilde{v}_k^++\lambda G_{0k}M_k^+\{u_j^-u_{k-j}^-+2u_j^-u_{k-j}^{+(0)}+u^{+(0)}u_{k-j}^{+(0)}-v_k^-v_{k-j}^--\tilde{v}_j^+\tilde{v}_{k-j}^++2\lambda u_j^-u_{k-j}^{+(1)}+2\lambda u_j^{+(0)}u_{k-j}^{+(1)}+\cdots\}. \tag{19}$$

Equating coefficients of each power of λ:

$$u_k^{+(0)}=\tilde{v}_k^+, \tag{20}$$

$$u_k^{+(1)}=G_{0k}M_k^+\{u_j^-u_{k-j}^--v_j^-v_{k-j}^-+2u_j^-u_{k-j}^{+(0)}\}, \tag{21}$$

where, from Eq. (20) there is a cancellation of $u^{+(0)}u^{+(0)}$ and $\tilde{v}^+\tilde{v}^+$ terms in Eq. (21). Also, from Eq. (14), when Eq. (21) is

conditionally averaged, the cancellation of u^-u^- and v^-v^- terms removes the triple nonlinearity to $O(\phi^-\phi^-)$. Accordingly the conditionally averaged coefficients are

$$\langle u_k^{+(0)}\rangle_c = \langle \tilde{v}_k^+\rangle_c = \langle \tilde{v}_k^+\rangle = 0, \tag{22}$$

$$\langle u_k^{+(1)}\rangle_c = G_{0k}M_k^+\langle \phi_j^-\phi_{k-j}^-\rangle_c. \tag{23}$$

Now substituting the perturbation series (17) into the equation for u^- [i.e., Eq. (10) in contracted notation and with the differential operator restored to the left-hand side] gives to order λ^2:

$$L_{0k}u_k^- = \lambda M_k^- u_j^- u_{k-j}^- + 2M_k^-\{\lambda u_j^- u_{k-j}^{+(0)} + \lambda^2 u_j^- u_{k-j}^{+(1)}\} + M_k^-\{\lambda u_j^{+(0)}u_{k-j}^{+(0)} + 2\lambda^2 u_j^{+(0)}u_{k-j}^{+(1)}\}. \tag{24}$$

Next, take the conditional average of both sides of Eq. (24). Two of the terms on the right-hand side vanish by Eqs. (22) and (23), while the term in $u^{+(0)}u^{+(0)}$ vanishes by homogeneity, leaving

$$L_{0k}v_k^- = \lambda M_k^- v_k^- v_{k-j}^- + 2\lambda^2 M_k^-\langle u_j^{+(0)}u_{k-j}^{+(1)}\rangle_c, \tag{25}$$

where from Eqs. (22) and (23) we see that the "cross term" in Eq. (24) is $O(\phi^-\phi^-)$ when conditionally averaged.

We shall now outline the procedure which leads to the second term on the right-hand side of Eq. (25) being written as

$$2\lambda^2 M_k^-\langle v^{+(0)}u_{k-j}^{+(1)}\rangle_c = \delta\nu_0(k)k^2u_k^- + O[\phi^-\phi^-, R^3(k_1)]. \tag{26}$$

The conditional average in Eq. (25) can be evaluated by forming a dynamical equation [26] for $\langle u_j^{+(0)}u_{k-j}^{+(1)}\rangle_c$. We have to bear in mind that both fields are taken at the same time t, so we need to form separate equations of motion for each of $u_j^{+(0)}$ and $u_{k-j}^{+(1)}$. Noting that $u_j^{+(0)}=\tilde{v}_j^+$, the evolution equation is just the model equation (15), while the evolution equation for $u_{k-j}^{+(1)}$ is obtained by rewriting Eq. (21) in terms of $k\to k-j$. The end result (and restoring full notation) is readily found to be

$$\langle u_\beta^{+(0)}(\mathbf{j},t)u_\gamma^{(1)+}(\mathbf{k}-\mathbf{j},t)\rangle_c = \int_{-\infty}^{t} ds e^{-(\nu_0 k^2+\nu_0|\mathbf{k}-\mathbf{j}|^2)(t-s)}2M_{\beta\delta\epsilon}(\mathbf{j}) \times \int d^3p\langle \tilde{v}_\epsilon(\mathbf{j}-\mathbf{p},s)\tilde{v}_\delta(\mathbf{k}-\mathbf{j},s)\rangle_c \mathbf{u}_\delta^-(\mathbf{p},s). \tag{27}$$

This result is substituted back into Eq. (25), where it is integrated over j in the band $k_1\leqslant j\leqslant k_0$. The crucial point here is that, for the vectors $\mathbf{j}-\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$ both in the range of asymptotic freedom of the model equation [Eq. (15)], the conditional average over the *model field* must in turn reduce to an unconditional covariance, with $\langle \tilde{v}_\epsilon(\mathbf{j}-\mathbf{p},s)\tilde{v}_\delta(\mathbf{k}-\mathbf{j},s)\rangle_c = P_{\epsilon\delta}(\mathbf{k}-\mathbf{j})\delta(\mathbf{k}-\mathbf{p})\tilde{Q}(|\mathbf{k}-\mathbf{j}|)$. As before, we argue that the *actual field* must be asymptotically free at the ultraviolet cutoff, and so we make the hypothesis that the model covariance can be related to the actual covariance by Taylor series expansion of the latter about $k=k_0$. This leads to our increment to the viscosity (in its current, corrected form [18]), here generalized to the nth shell elimination

$$\delta\nu_n(k) = \frac{1}{k^2}\int d^3j \frac{L(\mathbf{k},\mathbf{j})[\,Q^+(l)|_{l=k_n} + (l-k_n)(\partial Q/\partial l)|_{l=k_n}]}{\nu_n(j)j^2 + \nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2 - \nu_n(k)k^2}, \tag{28}$$

where $\mathbf{l}=\mathbf{k}-\mathbf{j}$ and the coefficient $L(\mathbf{k},\mathbf{j})$ is given by

$$L(\mathbf{k},\mathbf{j}) = -2M_{\alpha\beta\gamma}(\mathbf{k})M_{\beta\alpha\delta}(\mathbf{j})P_{\delta\gamma}(\mathbf{k}-\mathbf{j}). \tag{29}$$

Evidently, even the model field is not asymptotically free over the entire band, particularly in the neighborhood of $k=k_1$. However, an exact symmetry ensures that energy transfer vanishes for triads with all sides equal. Therefore there is a deficit of energy transfer from modes with wave numbers just below the cutoff, to those just above. Bearing in mind that the increment to the viscosity depends on an integral over the wave numbers in the band, it is perhaps not surprising that we obtain good results for the Kolmogorov prefactor, with the implication that the method captures energy transfer quite well.

Of course, it would be incorrect to assume that the same would apply for momentum transfers; for example, if we were to try to use Eq. (25) for large-eddy simulation. This can be seen by considering Eq. (10). If we form a correlation coefficient between u^- and the right-hand side of the equation, it must be equal to unity, indicating that u^- is in phase with the function on the right-hand side. However, correlation coefficients formed between u^- and the individual partitions of the right-hand side must all be less than unity, indicating that none of the individual partitions is in phase with u^-. This tells us that representation of any one of these individual partitions in terms of an eddy viscosity (or any deterministic quantity) acting on the u^- is actually incorrect; because a result such as Eq. (26) is necessarily perfectly in phase with u^-.

It is an article of faith in the application of RG that the final result (i.e., a prediction of an observable) should not depend on the values of arbitrarily chosen parameters; a concept sometimes referred to as renormalization invariance. In field theoretic RG, renormalization invariance is a defining postulate but in later applications to critical phenomena it has been seen as something which should emerge naturally from the method [2]. In the turbulence problem, the arbitrary parameters are the initial value of the viscosity ν_0 and the bandwidth η (or, equivalently, the spatial rescaling factor h).

In Fig. 3 we have seen that the renormalized viscosity is insensitive to the choice of initial viscosity and in Fig. 4 that the calculated Kolmogorov constant is insensitive over a range of values of the bandwidth. To be more specific, we find that $\alpha=1.62\pm0.05$ over the range $0.2\leqslant\eta\leqslant0.6$. This gives us some indication of the quantitative behavior of our approximation. Evidently this breaks down for $\eta<0.2$ because the band becomes so small that the integral over wave number j [in Eq. (28)] is dominated by behavior near the lower cutoff wave number k_n. Conversely, the breakdown of the theory for $\eta>0.6$ can be attributed to the inadequacy of

the first-order truncation of the Taylor-series expansion about the upper cutoff wave number k_{n-1}, when the bandwidth becomes large.

The perturbation expansion which we propose here differs in several respects from Gaussian perturbation theory. In the latter [10], all higher-order coefficients $u^{(1)},u^{(2)},\ldots$, are expressed in terms of $u^{(0)}$, which is Gaussian, and all averages are evaluated accordingly. In the present method we do not work with the coefficients explicitly, but instead with their conditional covariances $\langle u^{(0)}u^{(0)}\rangle_c$, $\langle u^{(0)}u^{(1)}\rangle_c,\ldots$.

We then solve governing equations for higher-order conditional covariances $\langle u^{(0)}u^{(1)}\rangle_c$, $\langle u^{(1)}u^{(1)}\rangle_c,\ldots$, and express these in terms of the zero-order $\langle u^{(0)}u^{(0)}\rangle_c$, which is then related to the actual turbulence covariance by means of the Taylor-series expansion about the upper cutoff wave number.

However, the need for a "different" perturbation theory is surely manifest if we compare the Gaussian result for $\delta\nu_0$ in Eq. (4), with its explicit dependence on the arbitrary stirring spectrum D_0k^{-y}, with the result from iterative conditional averaging, as given by Eq. (28), which depends on the actual turbulence spectrum at large wave numbers. Moreover, from Fig. 2, we can see that Gaussian perturbation theory is limited to low values of wave number where the coupling $R(k)\to 0$, as $k\to 0$. In contrast, the present method eliminates finite blocks of modes, in a process where the coupling increases initially but is ultimately bounded by $R(k^*)=0.4$ at the fixed point [17]. Another relevant order of magnitude can be obtained as follows. Choosing $k_{\max}=1.6k_d^{(0)}$ (a typical criterion for direct numerical simulations, with the prefactor chosen for easy arithmetic) and $\eta=h=0.5$, for a fixed point value of $n=4$ (see Fig. 3), and the numerical value of the fixed-point wave number is found to be $k_t=(0.5)^4 1.6k_d^{(0)}=0.1k_d^{(0)}$, in agreement with the usual value for the division between inertial and dissipation ranges.

Evidently considerations of this kind completely rule out the use of differentially small RG transformations when seeking to reduce the number of degrees of freedom of turbulence at large wave numbers. In order to eliminate large "blocks" of modes, as pointed out earlier, we have to evaluate a conditional average. Accordingly our definition of "soluble" in putting forward a soluble model, is one for which a conditional average reduces to an unconditional turbulence average. This brings us to the classic dilemma of many-body theory: faced with an insoluble problem one wishes to find an approximate model which is (a) reasonably representative of the exact problem and (b) soluble. It is well known that it is difficult to reconcile these two requirements, and perturbation theory is supposed to bridge the gap. Therefore, once we have settled for solubility, we are forced to a consideration of the question: how representative is the model?

This question is not easily answered. The model has been constructed by removing one class of interactions (i.e., the direct coupling or "cross" term) from the high-wave-number NSE. As the RG iteration begins in the far dissipation range (where viscous effects dominate), we might suppose that the absence of a part of the nonlinear term might not matter. However, when ultimately the calculation reaches a fixed point, it has entered the inertial range and we might wish the model to be compatible with the Kolmogorov spectrum, or, at least, to exhibit scale invariance and energy conservation; both of which underpin the $k^{-5/3}$ spectrum. This could be achieved by introducing the low-wave-number model field $\tilde{v}_k^-$ with its own evolution equation, which also has a class of interactions (in this case, the Reynolds-type $\tilde{v}_j^+\tilde{v}_{k-j}^+$ term) removed. This then raises the possibility of producing a unified treatment in which the low-wave-number model field is used to generate the subensemble for the conditional average. We are working on this at the present time and hope to submit it for publication in due course.

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[26] In Gaussian perturbation theory this can be done by direct substitution for $u_{k-j}^{+(1)}$ and evaluation of the Gaussian average in terms of the stirring forces. In our non-Gaussian theory, coefficients in the perturbation series bear an unknown relationship to the *actual* turbulence field and we have to use the re-expansion in the local Reynolds number to deal with both time dependence and the moment hierarchy.